United Kingdom
Mathematics Trust

Intermediate Mathematical Olympiad

Maclaurin paper

Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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1. A plank of wood has one end, A, against a vertical wall. Its other end, B, is on horizontal ground. When end A slips down 8cm, end B moves 4cm further away from the wall. When end A slips down a further 9cm, end B moves a further 3cm away from the wall. Find the length of the plank.

SOLUTION

Let the original distances from A and B to the corner where the ground meets the wall be a cm and b cm respectively.

Then, using Pythagoras' theorem, we have

$$a^2 + b^2 = (a - 8)^2 + (b + 4)^2 = (a - 17)^2 + (b + 7)^2$$

. After simplification, this produces the equations below.

$$2a - b = 10$$

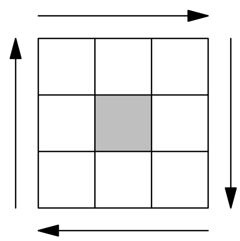
$$17a - 7b = 169$$

Solving these, we obtain $a = 33$ and $b = 56$.

The length of the plank is then 65 cm.

This problem relies on three Pythagorean triples (3, 4, 5), (5, 12, 13) and (33, 56, 65) where the first two are scaled up to give the same hypotenuse.

2. The digits 1 to 8 are placed into the cells of the grid on the right, making four three-digit numbers when read clockwise. For which values of k from 2 to 6 is it possible to create an arrangement such that all four of the three-digit numbers are multiples of k?



Solution

For $k = 2$, any arrangement with the four even numbers in the four corners will work.

For $k = 3$, there are plenty of possibilities. For example, the numbers 132, 285, 567 and 741 reading clockwise will work.

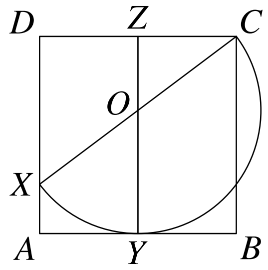
For $k = 4$, the corners would all have to be even to make the numbers even. In addition, the final two digits of each number would have to be divisible by 4. But neither 14, 34, 54 or 74 is divisible by 4. Hence this is impossible.

For $k = 5$, the last digit of each would need to be 5. As there is only one 5 available, this is impossible.

For $k = 6$, the corners would again have to be even. Moreover, the four digit sums would have to be multiples of 3. This would mean that $1 + 3 + 5 + 7 + 2(2 + 4 + 6 + 8)$ is a multiple of 3, but the sum is 56, which is not. Therefore this is also impossible.

3. $ABCD$ is a square and X is a point on the side DA such that the semicircle with diameter CX touches the side AB. Find the ratio $AX : XD$.

SOLUTION



Since we are interested in the ratio of sides, we can assign a side length for convenience as this would not affect the ratio of lengths. Let $CD = 2$.

Let O be the centre of the semicircle, let Y be the point of tangency on AB and let YO meet CD at Z.

Note that $OY = OX = OC$ as they are all radii. Since OY is perpendicular to AB, OZ is perpendicular to CD. Then Z is the midpoint of DC because triangles COZ and CXD are similar with scale factor 2.

Let $XD = x$, so $OZ = \frac{x}{2}$ and $OY = 2 - \frac{x}{2}$.

Hence $OC = 2 - \frac{x}{2}$ and, using Pythagoras' Theorem on triangle CZO, we have $1 + \left(\frac{x}{2}\right)^2 = \left(2 - \frac{x}{2}\right)^2$.

This leads to $x = \frac{3}{2}$, so $AX : XD = 1 : 3$.

ALTERNATIVE

As in the previous solution, we show that Y is the midpoint of AB.

$\angle XYC = 90°$ because it is the angle in a semi-circle.

Then, $\angle CYB = 90° - \angle XYA = \angle AXY$.

Hence $\triangle AXY \sim \triangle BYC$.

Therefore, $\frac{AX}{AY} = \frac{BY}{BC} = \frac{1}{2}$.

Now $AX = \frac{1}{2}AY = \frac{1}{2}BY = \frac{1}{4}BC = \frac{1}{4}AD$, so $AX : XD = 1 : 3$.

4. The ratio of the number of red beads on a chain to the number of yellow beads is the same as the ratio of the number of yellow beads to the number of blue beads. There are 30 more blue beads than red ones. How many red beads could be on the chain?

SOLUTION

Let the numbers of red, yellow and blue beads be r, y and b respectively.

Then we have $\frac{r}{y} = \frac{y}{b}$ and $b = 30 + r$ and we must find positive integer solutions.

We have $y^2 = br = r(30 + r) = (r + 15)^2 - 225$.

This rearranges to $(r + 15)^2 - y^2 = (r + 15 + y)(r + 15 - y) = 225 = 3^2 \times 5^2$, which we express as a product of positive integer factors m and n with $m > n$.

Then $r = \dfrac{m + n - 30}{2}$ and $y = \dfrac{m - n}{2}$.

The table shows all possible values solving the equation in positive integers and the respective values for r, y and b.

m	n	r	y	b
225	1	98	112	128
75	3	24	36	54
45	5	10	20	40
25	9	2	8	32
15	15	0	0	0

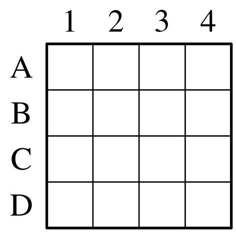
The final line of the table is not relevant as we do not have $m > n$ which leads to the ratio of red to yellow being indeterminate. However, it is useful to check we have considered all pairs of numbers multiplying to 225.

Hence the possible numbers of red beads are 2, 10, 24 or 98.

5. A 4 by 4 square is divided into sixteen unit cells. Each unit cell is coloured with one of four available colours, red, blue, green or yellow.
The 4 by 4 square contains nine different 2 by 2 “sub-squares". Suppose that we colour the sixteen unit cells in such a way that each 2 by 2 sub-square has one cell of each colour.
Prove that the four corner cells in the large 4 by 4 square must then be coloured differently.

SOLUTION

We will refer to squares by rows A, B, C and D and columns 1, 2, 3 and 4.



Suppose the grid can be filled with two corners the same colour.

Suppose adjacent corners, say A1 and A4, are the same colour, say green. Then A2 and B2 cannot be green because A1 is green. Also, A3 and B3 cannot be green because A4 is green.
But then the sub-square A2, A3, B2, B3 contains no green. This is a contradiction, so adjacent corners cannot be the same colour.

Suppose opposite corners, say A1 and D4, are the same colour, say green. Then one of C2 and B3 must be green to give the middle sub-square a green square. Since they create symmetrical diagrams, suppose, without loss of generality, B3 is green. Then C1 must be green so that the sub-square B1, B2, C1, C2 contains a green. Also, D2 must be green so that the sub-square C2, C3, D2, D3 contains a green. But then the sub-square C1, C2, D1, D2 contains two greens.

This is a contradiction, so opposite corners cannot be the same colour.

Since no colour can be in two corners, all four corners must contain different colours.

ALTERNATIVE

The whole grid has four 2 by 2 sub-squares so contains all four colours four times each.

Columns 2 and 3 have two sub-squares, so contain two of each colour. The centre sub-square has one of each colour, so A2, A3, D2 and D3 contain one of each colour.

Rows B and C have two sub-squares, so contain two of each colour.

Therefore the twelve squares excluding the corners contain three of each colour.

Therefore the four corners must contain one of each colour once each, so they are all different colours.

ALTERNATIVE

The whole grid has four 2 by 2 sub-squares so contains all four colours four times each.

Each corner square is in one sub-square. Each edge square is in two sub-squares. Each centre square is in four sub-squares. Each colour needs to be in 9 sub-squares.

The total for each colour must be 9, which is odd, so must contain an odd number of corner squares. Therefore all four colours need to be in at least one corner square. There are only four corners, so all four corners contain different colours.

6. Let m, n be fixed positive integers. Prove that there are infinitely many triples of positive integers (x, y, z) such that

$$x^{mn+1} = y^m + z^n$$

for each pair of values (m, n).

SOLUTION

We notice that one such triple is $x = 2$, $y = 2^n$ and $z = 2^m$ since $y^m + z^n = 2^{mn} + 2^{nm} = 2^{mn+1}$.

We now need to generalise this so that there are infinitely many triples. For this we introduce a variable k which takes positive integer values. The triple above will represent the case $k = 0$.

We wish to define functions $f(m, n)$ and $g(m, n)$ so that $y = 2^{f(m,n)}$ and $z = 2^{g(m,n)}$.

Then $y^m = 2^{mf(m,n)}$ and $z^n = 2^{ng(m,n)}$ and we need to ensure that $mf(m, n) = ng(m, n)$ and $mf(m, n) + 1$ has a factor of $mn + 1$ for all values of k.

One way to do this is to take $f(m, n) = n + nk(mn + 1)$ and $g(m, n) = m + mk(mn + 1)$.

Now we have a common index of $mn + mnk(mn + 1)$ for both y^m and z^n, and when we add them we have $2^{1+mn+mnk(mn+1)}$.

Now, $1 + mn + mnk(mn + 1) = (mnk + 1)(mn + 1)$.

That means $2^{1+mn+mnk(mn+1)} = 2^{(kmn+1)(mn+1)}$.

When $x = 2^{kmn+1}$, $y = 2^{n(kmn+k+1)}$ and $z = 2^{m(kmn+k+1)}$, we obtain $x^{mn+1} = y^m + z^n$ as desired.

Since k can be any non-negative integer, we have infinitely many triples of positive integers for each pair of values (m, n).

ALTERNATIVE

We notice that one such triple is $x = 2$, $y = 2^n$ and $z = 2^m$ since $y^m + z^n = 2^{mn} + 2^{nm} = 2^{mn+1}$.

We notice that if $y = a^n$ and $z = a^m$ so that $y^m + z^n = a^{mn} + a^{nm} = 2a^{mn}$. We want that when we collect terms we get an extra factor of a instead of a factor of 2.

We consider $y = ka^n$ so that $y^m + z^n = k^m a^{mn} + a^{nm} = (k^m + 1)a^{mn}$. If we let $a = k^m + 1$ then $y^m + z^n = a^{mn+1}$ and $x = a$ is a solution.

Since k can be any positive integer, we have infinitely many triples of positive integers for each pair of values (m, n).

Note that this can be generalised further by letting $y = pa^n$ and $z = qa^m$ where $a = p^m + q^n$ and $x = a$ to give infinitely many solutions.